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APPLICATION OF NON-SELF-ADJOINT OPERATOR THEORY TO THE SINGULAR-- TC(U)

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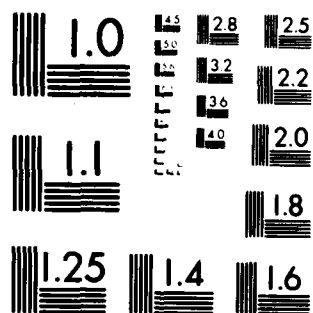
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(18) AFOSR TR. 80-0689

Final Report (15) F49620-79-C-0128

LEVEL II

(6) APPLICATION OF NON-SELF-ADJOINT OPERATOR THEORY TO THE SINGULARITY EXPANSION METHOD (SEM) AND THE EIGENMODE EXPANSION METHOD (EEM) IN ACOUSTIC AND ELECTROMAGNETIC PROBLEMS.

(12) 233  
(11) Jul 80  
(16) 2304  
(17) A4  
(10) Charles L. Dolph  
Department of Mathematics  
The University of Michigan  
Ann Arbor, Michigan 48109  
228  
715

July 1980

(9) Final Rept. 1 Jun 79-31 May 80

Final Report for the Period: 1 June 1979 to 31 May 1980

Approved for Public Release  
Distribution Unlimited

Prepared for

Air Force Office of Scientific Research  
Building 410, Bolling Air Force Base  
Washington, D.C. 20332

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REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM	
1. REPORT NUMBER <b>AFOSR-TR- 80-0689</b>	2. GOVT ACCESSION NO. <b>AD-A081</b>	3. RECIPIENT'S CATALOG NUMBER <b>747</b>	
4. TITLE (and Subtitle) <b>APPLICATION OF NON-SELF-ADJOINT OPERATOR THEORY TO THE SINGULARITY EXPANSION METHOD (SEM) AND THE EIGENMODE EXPANSION METHOD (EEM) IN ACOUSTIC AND ELECTROMAGNETIC PROBLEMS</b>		5. TYPE OF REPORT & PERIOD COVERED <b>Final</b>	
		6. PERFORMING ORG. REPORT NUMBER	
7. AUTHOR(s)  <b>Charles L. Dolph</b>		8. CONTRACT OR GRANT NUMBER(s)  <b>F49620-79-C-0128</b>	
9. PERFORMING ORGANIZATION NAME AND ADDRESS <b>Department of Mathematics University of Michigan Ann Arbor, Michigan 48109</b>		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS  <b>61102F 2304/A4</b>	
11. CONTROLLING OFFICE NAME AND ADDRESS <b>Air Force Office of Scientific Research Building 410, Bolling Air Force Base Washington, D.C. 20332</b>		12. REPORT DATE <b>July 1980</b>	
		13. NUMBER OF PAGES <b>19</b>	
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report)  <b>UNCLASSIFIED</b>	
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE	
16. DISTRIBUTION STATEMENT (of this Report)  <b>Approved for public release. Distribution unlimited.</b>			
17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)			
18. SUPPLEMENTARY NOTES			
19. KEY WORDS (Continue on reverse side if necessary and identify by block number)  <b>Nonselfadjoint Operator Theory; Singularity Expansion Method (SEM); Eigenmode Expansion Method (EEM); Dyadic Green's Functions</b>			
20. ABSTRACT (Continue on reverse side if necessary and identify by block number)  <b>➤ Eigenfunction expansions for nonselfadjoint operators are important for scalar and electromagnetic wave scattering. Two methods: the Singularity Expansion Method (SEM), and the Eigenmode Expansion Method (EEM) which had been developed separately were studied. Criteria for their validity were</b>  <b>(over)</b>			

*Unclassified*

20. Abstract. (continued)

*X* established; moreover, the poles of the Green's function of the SEM are in 1-1 correspondence with the zeros of the eigenvalues of the EEM. A constructive numerical process for determining the poles of the Green's function was developed. Among several other results was the establishment of variational principles for the spectrum of compact nonselfadjoint operators.

Another research area was the singularity behavior of eigenfunction expansions of various Green's functions in electromagnetic theory. The principal result shows that in an eigenfunction of a typical Green's function the point singularity of a Green's function is represented by a layer of surface singularity. This characteristic is analogous to the Gibbs' phenomenon where the representation of a discontinuous function by an orthogonal expansion creates the spikes which are absent in the original function.

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## Technical Narrative

Charles L. Dolph and Alexander G. Ramm

### 1. Scientific Area, Central Objectives, Main Approach to the Problem.

Scalar and electromagnetic wave scattering is of great interest in physics and technology. In the open system theory in the framework of the traditional approach one should deal with operators with continuous spectrum. From the computational point of view it is much more convenient to deal with operators with discrete spectrum. Furthermore in many practical problems the systems have resonant properties so that only one or two terms in an appropriate representation of solutions to scattering problems are of importance. These ideas were extensively studied in the physics and engineering literature [2-5]. Engineering study was initiated by Kacnelenbaum (1968) [2] in the USSR and by C. Baum (1973) [3] in the USA. Mathematical study was initiated by Ramm (1969) [4]; a review paper which includes a discussion of the SEM is Dolph-Scott [5]. The ideas are used in any scientific area where wave scattering is under study. The objective of the theory is to give a computational approach to scattering problems which uses eigenfunction expansions (of nonclassical type) for those operators with discrete spectrum. These operators are nonselfadjoint in most cases, because in open systems there are some losses (radiation losses etc.) which lead to dissipation. Therefore, from the mathematical point of view, the scientific area is the spectral theory of nonselfadjoint operators and the objective is the study of eigenfunction expansions for nonselfadjoint operators. These questions have been insufficiently studied in the literature [1]. For example, nothing is known about eigenfunction expansions for the basic operator of laser theory:

$$Af = \int_{-1}^1 \exp i(x-y)^2 f(y) dy$$

From both the mathematical and physical points of view a very natural approach to the above problems consists of the use of integral or, more generally, pseudo-differential equations on the boundary of the scatterer. This approach requires the study of spectral properties of the corresponding nonselfadjoint operators.

## 2. Air Force Relevance of the Research.

While such relevance is already clear from the nature of the problems mentioned in Section 1, it is worthwhile to point out that the Singularity Expansion Method (SEM) and the Eigenmode Expansion Method (EEM) can both be considered as examples of the general theory mentioned in Section 1. Also, the problem of target identification which consists of the identification of the obstacle (target) from the transient field scattered by this obstacle, can be treated within the framework of the general theory.

## 3. Main Results.

The following results were obtained during the contractual period:

- 1) A criterion for the validity of the SEM and EEM was established.
- 2) The basic relation between the SEM and the EEM was given in that the poles of the Green's function of the SEM were shown to be in one-to-one correspondence with the zeros of the eigenvalues of the EEM.
- 3) A constructive numerical process for determining the poles of the Green's function was developed.
- 4) The continuous dependence on the poles was established for a class of boundary perturbations.
- 5) Sufficient conditions for the root vectors in the EEM to form a Riesz basis [3] and a Riesz basis with brackets [2] have been found.
- 6) Variational principles for the spectrum of compact nonselfadjoint operators have been established.

We do not formulate these results in detail because they were reported to AFOSR earlier ([6] - [10]) and presented in the series of talks listed in the Appendix.

## 4. Open Problems, Directions for Future Research.

There are many open problems in this field. Many of them are of immediate practical interest. We mention some of the open problems.

- 1) To what extent do the complex poles of Green's functions determine the shape of the body?



- 2) To what extent do purely imaginary poles of Green's functions determine the shape of the body?
- 3) When is it more convenient to use the single layer potential for field representations rather than the double layer potential?
- 4) What is the short wave asymptotic distribution of the complex poles?
- 5) What stationary principles exist for the complex poles?
- 6) What is the relation between multiplicities of the complex poles and of the zeros of the eigenvalues of the corresponding integral equations?
- 7) To prove that the root system of the integral operators in scattering theory forms a Riesz basis without brackets.
- 8) Whether the complex poles of Green's functions for convex lossless bodies are simple?
- 9) Consideration of electromagnetic scattering problems.

5. Suggestions.

It would be interesting and useful to organize some numerical experiments (e.g., calculation of the complex poles) for a practical evaluation of some of the algorithms obtained in the course of this research. To this end a couple of graduate students and a postdoctoral fellow could be incorporated.

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#### Appendix I

Talks on Nonselfadjoint Operators and Scattering Theory Given by A. G. Ramm

Symposium on Scattering	June 1979	Columbus, Ohio
Colloquium	September 1979	University of Delaware
Colloquium	September 1979	Iowa State University
Symposium on Ill-Posed Problems	October 1979	Newark, Delaware
Colloquium and Workshop	October 1979	Naval Research Laboratory
Technical Report	October 1979	Mathematics Research Center University of Wisconsin
American Mathematical Society,	November 1979	Kent State University
Colloquium	November 1979	Princeton University
Colloquium	November 1979	Cornell University
Seminar Talk	November 1979	University of Michigan
Colloquium and Talk for Geophysics Dept.	November 1979	University of Utah
Seminar Talk	November 1979	University of Michigan
Colloquium	December 1979	Texas A & M
Seminar Talk	January 1980	University of Kansas

Colloquium	February 1980	Brown University
Colloquium	March 1980	Argonne Laboratory
Organizer of Special Session American Mathematical Society	April 1980	Indiana University
Special Lectures	May 1980	University of California at San Diego and Berkeley
Special Lectures	May 1980	Stanford University

## Appendix II

Invited Talk: "Nonselfadjoint operators in diffraction and scattering"  
by C. L. Dolph and A. G. Ramm; presented at Meeting of  
American Mathematical Society, Boulder, Colorado, March 1980

### NONSELFADJOINT OPERATORS IN DIFFRACTION AND SCATTERING

C. L. Dolph and A. G. Ramm

Department of Mathematics  
University of Michigan  
Ann Arbor, MI 48109

1. In many problems of interest for physicists and engineers non-selfadjoint operators arise naturally. For example the EEM (eigenmode expansion method) can be described as follows. Let  $\Omega$  be an exterior domain with a smooth closed compact boundary  $\Gamma$ ,

$$\begin{aligned} (\Delta + k^2)u = 0 \quad \text{in } \Omega, \quad u|_{\Gamma} = f, \quad |x| \left( \frac{\partial u}{\partial |x|} - iku \right) \rightarrow 0 \\ \text{as } |x| \rightarrow \infty, \quad k^2 > 0. \end{aligned} \quad (1)$$

Let us look for a solution of (1) in the form

$$u = \int_{\Gamma} \sigma(t) \exp(ikr_{xt}) (4\pi r_{xt})^{-1} dt, \quad r_{xt} = |x - t|. \quad (2)$$

For the unknown  $\sigma$  we get equation

$$A\sigma \equiv \int_{\Gamma} \exp(ikr_{st}) (4\pi r_{st})^{-1} \sigma(t) dt = f(x). \quad (3)$$

Operator  $A$  is nonselfadjoint in  $L_2(\Gamma)$ . Suppose that its root system forms a basis of  $H = L_2^2(\Gamma)$ . Then we can look for a solution of (3) in the form of the series, expanding  $\sigma(t)$  and  $f(x)$  according to the root vectors of  $A(k)$ . This is called EEM for solution of (1). Thus we have question 1) Does the root system of  $A(k)$  form a basis of  $H$ ?

It is easy to prove that the Green function  $G(x, y, k)$  of the exterior Dirichlet (or Neumann) problem can be meromorphically

continued on the whole complex plane  $k$  and its poles  $k$  lie in the lower half-plane  $\text{Im } k < 0$ . If  $f$  is a smooth function with compact support,  $v = \int G(x, y, k) f(y) dy$ ,  $\int = \int_{\Omega}$  and

$$(*) |v| \leq C(1 + |k|^a)^{-1}, \quad a > 0.5, \quad C = \text{const.}, \quad C = C(\text{Im } k),$$

then the solution of the problem

$$u_{tt} = \Delta u \quad \text{in } \Omega, \quad t > 0, \quad u|_{\Gamma} = 0, \quad u|_{t=0} = 0, \quad u_t|_{t=0} = f(x) \quad (4)$$

can be represented in the form

$$u(x, t) = \sum_{j=1}^n \exp(-ik_j t) v_j(x, t) + o(\exp\{-|\text{Im } k_n| t\}), \quad t \rightarrow +\infty \quad (5)$$

where  $v_j(x, t)$  grow not faster than  $t^m$  as  $t \rightarrow +\infty$  and  $m$  is some integer. Expansion (5) is an example of SEM (singularity expansion method). This leads to questions: 2) What can be said about location of the poles  $k_j$ ?; 3) When is (\*) valid?; 4) to what extent does the set  $\{k_j\}$  determine the obstacle?; 5) how can the  $k_j$  be calculated?; 6) whether the poles  $k_j$  are simple?

2. The answer to question 1) is given in [1] and is described below. In [2]-[4] some results about bases with brackets are given. Some answers to question 2) are given in [5]-[7]. Answers to question 3) are given in [8]-[10]. Answer to question 4) is unknown. Answer to question 5) is given in [11], [12]. Answer to 6) is unknown, but some engineers (C.E. Baum e.g.) think that if  $\Gamma$  is convex then the Green function of the exterior Dirichlet problem has simple poles. Some particular cases when this is true are discussed in [13]. In [14], a survey of the SEM is given and [15] presents an engineering point of view. In [16] a survey of what is known about questions 1)-6) is given and in [17] some relevant results can be found. In [18] the relation between SEM and the mathematical scattering theory is discussed. In [19], [20] variational principles are discussed for nonselfadjoint problems.

3. In this section we answer question 1). If  $A = L + T$ , where  $L$  is a selfadjoint operator on a Hilbert space  $H$ , the spectrum  $\{\lambda_j\}$  of  $L$  is discrete,  $\lambda_n = cn^p(1 + O(n^{-\delta}))$  where  $c, p, \delta$  are

some positive constants, and  $|Tf| \leq c_1 |L^a f|$ ,  $c_1 > 0$ ,  $a < 1$  for all  $f \in D(L^a)$ ,  $D(L^a) \subset D(T)$ , then: 1) the root system of  $A$  forms a Riesz basis of  $H$  ( $A \in R$ ) if  $\delta > 1$  and  $p(1-a) > 2$ ; 2) if  $\delta > 0$ ,  $p(1-a) > 1$  it forms a Riesz basis of  $H$  with brackets, ( $A \in R_b$ ); 3) if  $\lambda_n^{1+a}(\lambda_{n+1} - \lambda_n)^2 \rightarrow 0$  as  $n \rightarrow \infty$ , then  $A \in R$ . Let us give the definition of the Riesz basis with brackets. Let the system  $\{f_j\}$  form an orthonormal basis of  $H$ ,  $m_1 < m_2 < \dots < m_n < \dots \rightarrow \infty$  is a sequence of integers,  $\{F_j\}$  is the sequence of the subspaces, where  $F_j$  is the linear span of  $f_{m_j}, f_{m_j+1}, \dots, f_{m_{j+1}-1}$ . Let  $\{h_j\}$  be a minimal and complete system in  $H$ ,  $H_j$  is the linear span of  $h_{m_j}, \dots, h_{m_{j+1}-1}$ . If there exists a map  $B \in L(H)$ ,  $BH_j = F_j$ ,  $j = 1, 2, \dots$ , then the system  $\{h_j\}$  is called a Riesz basis of  $H$  with brackets and the numbers  $m_j$  define the bracketing. By  $L(H)$  we denote the set of linear bounded invertible operators which map  $H$  onto  $H$ .

#### 4. Problems.

1. Find an asymptotic formula for  $\tilde{k}_j$  as  $|\tilde{k}_j| \rightarrow \infty$ , where  $\{\tilde{k}_j\}$  are the poles with minimal imaginary parts. For purely imaginary poles some information about asymptotic distribution is given in [7].

2. To what extent does the set  $\{k_j\}$  determine the obstacle?

3. Is it true that the complex poles of the Green function of the exterior Dirichlet problem are simple provided that  $\Gamma$  is convex?

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### Appendix III

Other Papers Written by A. G. Ramm During the Period: 1 June 1979 - 31 May 1980

1. Electromagnetic wave scattering by small bodies. Proc. International Symposium on Scattering and T-Matrix, Columbus, Ohio, June 1979, Pergamon Press, N.Y. (1980).
2. Investigation of a class of systems of integral equations, J. Math. Anal. Appl. (1980).
3. Perturbations preserving asymptotic of spectrum of linear operators, J. Math. Anal. Appl. (1980).
4. Uniqueness theorem of use in general relativity (with P. Mishnaevsky), J. Math. Anal. Appl. (1980).
5. On the quasistatic boundary value problem of electrodynamics (with C. L. Dolph), J. Math. Anal. Appl. (1980).
6. Two sided estimates of the scattering amplitude at low energies, to appear in J. Math. Physics, (1980).
7. Stable solutions of some ill-posed problems, submitted to Tech. Reports of Naval Research Laboratory, (1980).
8. A new proof of the absence of a positive discrete spectrum of the Schrödinger operator (with B. A. Taylor), J. Math. Physics, (1980).



## Technical Narrative

C. T. Tai

During the contract period I have directed my effort to the study of the singularity behavior of the eigenfunction expansions of various Green's functions in electromagnetic theory; they include the Green's function for Poisson's equation and the dyadic Green's function for Maxwell's equations. The discontinuous behavior of the eigenfunction expansion of these functions is discussed with the aid of equivalent layers of surface charge, current sheet, and surface polarization. The principal result shows that in an eigenfunction of a typical Green's function the point singularity of a Green's function is represented equivalently by a layer of surface singularity. This characteristic is analogous to the Gibbs' phenomenon where the representation of a discontinuous function by an orthogonal expansion (Fourier series) creates the spikes which is absent in the original function. The only difference is that in our case the surface discontinuity has replaced the point singularity in the eigenfunction expansion.

In the case of the dyadic Green's function we have examined in detail the function pertaining to a rectangular waveguide. Similar studies are currently being done for functions associated with other diffracting bodies.

A paper, "Equivalent Layers of Surface Charge, Current Sheet and Polarization in the Eigenfunction Expansions of Green's Functions in Electromagnetic Theory", based on this research has been written. It has been submitted to the IEEE Transaction on Antennas and Propagation for publication.

It is my hope that this research will be continuously supported by the Air Force Office of Scientific Research so the singularity behavior of various Green's function in electromagnetic theory will be better understood.

A communication has been written to comment on the recent paper by A. D. Yaghjian ("Electrical dyadic Green's functions in the source region", Proc. IEEE, Vol. 68, No. 2, pp. 248-263, February, 1980). A copy of the communication, which has been accepted by IEEE for publication, is also attached to this report. It seems clear that much of the confusion created by that article is due to the lack of understanding of the singularity behavior of dyadic Green's functions. For this reason, it is desirable that further work should be done to educate the public as well as to consolidate the concept and the foundation of this discipline.

# ELECTRIC DYADIC GREEN'S FUNCTIONS IN SOURCE REGION\*

C. T. Tai

Radiation Laboratory  
The University of Michigan  
Ann Arbor, MI 48109

The author of a recent article [1] bearing the above title has criticized severely the derivation and interpretation of the well established formula

$$\mathbf{E}(\mathbf{R}) = i\omega\mu_0 \int_{V_J} \bar{\mathbf{G}}_e \cdot \mathbf{J} dv' \quad (1)$$

obtained by the delta function technique. Using the classical function technique he has derived a formula of the form

$$\mathbf{E}(\mathbf{R}) = i\omega\mu_0 \lim_{V_\delta \rightarrow 0} \int_{V_J - V_\delta} \bar{\mathbf{G}}_e^0 \cdot \mathbf{J} dv' + \frac{1}{i\omega\epsilon_0} \bar{\mathbf{L}} \cdot \mathbf{J} \quad (2)$$

where  $\bar{\mathbf{G}}_e^0$  denotes what he calls the conventional electric dyadic Green's function and  $\bar{\mathbf{L}}$  is the source dyadic. There are several ambiguities in his paper that we would like to call attention to.

For convenience the function  $\bar{\mathbf{G}}_e$  in (1) will be designated as the standard electric dyadic Green's function as distinct from  $\bar{\mathbf{G}}_e^0$  in (2). Some comments on the characterization of  $\bar{\mathbf{G}}_e^0$  as described by that author will be given later. If  $\bar{\mathbf{G}}_e^0$  in (2) is replaced by  $\bar{\mathbf{G}}_e$  then the equation represents the so-called regularized version of

(1) as first discussed by Van Bladel [2] who only considers a principal volume,  $V_\delta$ , in the shape of a sphere. In 1972 this correspondent, with help from Dr. Olov Einarsson, also studied the source dyadic for bodies of different shape. Although the work was not published the information was passed verbally to various people including the author of [1].

What we have done was to demonstrate the identity

$$\lim_{V_\delta \rightarrow 0} \int [\nabla \times \nabla \times \bar{\bar{G}}_e - k^2 \bar{\bar{G}}_e] dv = \bar{\bar{I}} \quad (3)$$

for several different shapes,  $V_\delta$ , including spheres and cylinders. Equation (3), of course, is simply a statement of the behavior of  $\bar{\bar{G}}_e$  in a neighborhood of the source point, and this boundary condition is implied when one writes the differential equation for  $\bar{\bar{G}}_e$  in the form

$$\nabla \times \nabla \times \bar{\bar{G}}_e - k^2 \bar{\bar{G}}_e = \bar{\bar{I}}\delta(\bar{R} - \bar{R}') \quad (4)$$

using the delta function together with the idemfactor as the inhomogeneous part of that differential equation. The derivation of (1) based on the delta function technique merely imposes this characterization in an elegant way as first shown by Levine and Schwinger [3]. The formula itself without using the dyadic notation can also be derived by the method of potentials. It is not clear to this correspondent why the author of [1] considers (1) to be an ill-conditioned solution, particularly in view of the existence theorem

for this integral discussed by Fikioris [4]. On the other hand, if one accepts (2) as the rigorous solution with  $\bar{G}_e^0$  replaced by  $\bar{G}_e$  then it certainly can be written in the form of (1) since, by definition

$$\lim_{V_\delta \rightarrow 0} \int_{V_\delta} \bar{G}_e \cdot J \, dv' = -\frac{1}{k^2} \bar{L} \cdot J. \quad (5)$$

It should be mentioned that for numerical calculations of  $\bar{E}(\bar{R})$  involving a free space Green's function the regularization has to be applied to finite cells. In this case (1) can be separated into two parts:

$$\bar{E}(\bar{R}) = i\omega\mu_0 \int_{V_j - V_\delta} \bar{G}_e \cdot \bar{J} \, dv' + \int_{V_\delta} \bar{G}_e \cdot J \, dv' \quad (6)$$

The value of the second integral in (6) now depends not only on the shape of  $V_\delta$  but also on its size. In a yet unpublished work [5] this aspect of the problem is considered in detail but the topic is outside the scope of the present note. What we want to point out is that the regularization with an infinitesimal  $V_\delta$  merely gives an interpretation of (1), and in numerical work it is the finitely regularized version (6) that must be used.

One of the most confusing aspects of [1] deals with the characterization of  $\bar{G}_e^0$ . According to [1],  $\bar{G}_e^0$  satisfies the following equations:

$$\bar{G}_e^0 = \frac{1}{k^2} \left[ \nabla \times \bar{G}_m^0 - \bar{L}_\delta(\bar{R} - \bar{R}') \right], \quad (7)$$

$$\nabla \times \bar{G}_e^0 = \bar{G}_m^0. \quad (8)$$

By eliminating  $\bar{G}_m^0$  from (7) and (8), one finds

$$\nabla \times \nabla \times \bar{G}_e^0 - k^2 \bar{G}_e^0 = \bar{L} \delta(\bar{R} - \bar{R}') \quad (9)$$

The above equations, of course, is contradictory to (4), and the solution to (9) is not what the author of [1] used in deriving (2), nor in illustrating its application to the waveguide problem. It is well known that a Green's function, scalar, vector, or dyadic, is an entire entity. There is nothing like conventional or unconventional. It has a unique and pre-specified behavior in the neighborhood of the singularity, and therefore automatically satisfies a condition such as the one described by (3). It is, perhaps, the mishandling of this aspect of the theory that leads the author of [1] to create an unorthodox 'equation' like (7).

We consider now the application of (1) and (2), with  $\bar{G}_e^0$  replaced by  $\bar{G}_e$  in the latter formula, to find  $\bar{E}(\bar{R})$  inside a waveguide. The explicit expression for  $\bar{G}_e$  is known to have the following form [6]

$$\bar{G}_e = \bar{S}^{\pm} - \frac{1}{k^2} \hat{z} \hat{z} \delta(\bar{R} - \bar{R}') \quad (10)$$

where the series  $\bar{S}^+$  applies to  $z > z'$ , the series  $\bar{S}^-$  to  $z < z'$ , and the source point is located in the plane  $z = z'$ . The series  $\bar{S}^{\pm}$  is

discontinuous at  $z = z'$  and it is also singular at  $\bar{R} = \bar{R}'$ . We assume a current distribution which has a  $z$  component only. Substituting (10) into (1) we obtain

$$\bar{E}(\bar{R}) = i\omega\mu_0 \int_{V_J} \bar{S}^{\pm} \cdot \bar{J} dv' + \frac{1}{i\omega\epsilon_0} J_z \hat{z} \quad (11)$$

In evaluating the volume integral we use  $\bar{S}^+$  for  $z > z'$  and  $\bar{S}^-$  for  $z < z'$ . The author of [1] illustrates the application of (2) using a principal volume in the shape of a needle. He uses our  $\bar{S}^{\pm}$  as his  $\bar{G}_e^0$ . The result yields

$$\begin{aligned} \bar{E}(\bar{R}) &= i\omega\mu_0 \lim_{V_{\delta} \rightarrow 0} \int_{V_J - V_{\delta}} \bar{S}^{\pm} \cdot \bar{J} dv' \\ &= i\omega\mu_0 \left[ \int_{V_J} \bar{S}^{\pm} \cdot \bar{J} dv' - \lim_{V_{\delta} \rightarrow 0} \int \bar{S}^{\pm} \cdot \bar{J} dv' \right] \quad (12) \end{aligned}$$

The integrals are not carried out in the referenced paper, but in a private communication to this correspondent Professor Robert E. Collin has proved analytically that the second integral in (12) is indeed equal to  $J_z \hat{z} / (i\omega\epsilon_0)$  for a typical distribution of  $J_z$  in a rectangular waveguide. The proof for an arbitrary  $\bar{J}$  can be done accordingly, though it is by no means a simple exercise. In fact, this correspondent failed to recognize the identity until it was pointed out to him by Professor Collin. From this example it is clear that there is no necessity to use the regularized version of (1) to find  $\bar{E}(\bar{R})$  in a waveguide. Unlike free space problems a direct application of (1)

and (10) avoids the tedious calculation of the explicit term  $J_z \hat{z} / (i\omega\epsilon_0)$  when the shape of the principal volume is not wisely chosen. Of course, if a principal volume in the shape of a pill-box is chosen (2) also yields immediately (11). It should be emphasized again that one must use the standard  $\bar{G}_e$  in (2) and not the one described by the solution of (9).

There are many other statements in [1] which are not comprehensible to this correspondent. For example, there is absolutely no discrepancy between the work of Tai and Ro enfeld and that of Rahmat-Samii.

The author appreciates the many valuable discussions on this subject with Professor Robert E. Collin. The kindness of Professor S. W. Lee for sending a prepublication copy of Reference [5] is gratefully acknowledged. Professor T.B.A. Senior made valuable suggestions in regard to the preparation of this communication. The comments from the author's graduate students, particularly Messrs. M. Naor and S. Giles, were very helpful in the presentation of this work.



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